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Griffith cracks in the mesoscopic microcrack theory

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Abstract
The mesoscopic concept is applied to the description of microcracked brittle materials. The mesoscopic equations are solved in a special case when the microcracks are developing according to the Rice–Griffith evolution law. The evolution of the crack distribution function is investigated in the case of simple loading conditions and for two different initial crack distribution functions. The time dependence of the average crack length is also calculated.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

There are several different approaches for the description of the behaviour of material with complex microstructure, e.g., granular media, liquid crystals or damaged solid materials. One of the basic approaches is the macroscopic one applying thermodynamic concepts and principles to describe the microstructure (e.g., internal thermodynamic variables). The other basic but different approach is statistical, where the microstructure is considered as a statistical ensemble and the macroscopic variables are calculated from the interactions between the microstructural elements using the principles of statistical mechanics. Modelling the microstructure tests the limits of applicability of both theories and therefore the basic principles of our understanding regarding the material behaviour.

The microscopic elements of statistical cracking phenomena are the single cracks. The form and validity of single crack evolution equations is an important problem in statistical microcracking. Beyond the well-known classical results (see, e.g., [1, 2]) there are new and

promising approaches and investigations in this field that should be considered. Langer and his co-workers [3–5] gave a microscopic background of the internal variable approach of plasticity [6]. Using molecular dynamic simulations they found some good microscopic candidates for the internal variables. Based on these observations, a so-called *shear deformation zone* (STZ) model was established, suggesting a dynamical equation of these internal variables, including also special unilateral requirements. Several inconsistencies of classical single crack models can be explained using this theory. Another interesting approach to single crack phenomena considers cracking as a kind of nucleation problem where elastic and conservation effects are included in the nonlocal (coarse grained) phase field model [7–9], or the elastic effects are included in a more direct nucleation approach [10–12]. Both STZ and the phase field approaches can be considered as phenomenological thermodynamic theories.

On the other hand, partially independently of the previous considerations, microcracking is also treated from a statistical point of view. There are two competing models in this respect. Sornette and his co-workers consider cracking as a critical phenomenon where rupture is a critical point (see, e.g., [13]). Their approach is supported by experimental scaling observations and by numerical simulations [14, 15]. However, other experimental observations indicate that failure is not a simple critical phenomenon [16] and the unilaterality (intrinsic irreversibility) of microdynamics is not negligible. Other observations and simulations suggest that the rupture of the material is better considered as a first-order phase transition and the cracking as spinodal nucleation [17]. Scaling calculations based on the previously mentioned nucleation models partially support that approach.

From the point of view of nonequilibrium thermodynamics the background of both the statistical and single crack models is the violation of thermodynamic stability in the nonequilibrium state space. This observation alone can unify several classical failure and fracture criteria and explain experimental observations [18, 19].

However, between pure phenomenological and statistical approaches there is an independent family of theoretical descriptions that we can call *hypercontinuum theories*. In these theories the physical quantities are not simple macroscopic fields given on the appropriate spacetime (classical, special or general relativistic), but one assumes that in addition to the spacetime variables they depend also on some mesoscopic variables characterizing the microstructure of materials. These theories have almost as long a history as the traditional macroscopic and microscopic ones. At the beginning they were used to model rotating internal particles in microcrystalline materials (see the basic works of the Cosserat brothers and Mindlin [20, 21]); nowadays generalized continua are used to model, e.g., granular media [22–24], liquid crystals [25, 26], ferrofluids [27, 28], polymer solutions [29, 30], but one can meet hypercontinuum in general relativity, too [31, 32] (see also [33] and the references therein). That model family is capable of incorporating both statistical and macroscopic observations.

Hypercontinuum theories represent models of the material where one can get information from below the macroscopic but above the microscopic level. This gives the basic motivation and modelling idea for the so-called mesoscopic theory, which uses both macroscopic, continuum and microscopic, statistical concepts constructing the basic equations of motion for the hypercontinuum. This approach was successfully applied for the description of liquid crystals (see [34] and the references therein).

Recently a mesoscopic theory was suggested for a special class of damaged materials, for microcracked brittle solid media [35]. In this paper we will see that considering the peculiar properties of microcracking simplifies the mesoscopic equations well enough to make them solvable with relatively little effort. This simplification results in a generalized Liouville equation that we will solve considering a simple evolution equation for the microcracks in the case of two different initial crack distribution functions. We applied Rice–Griffith microscopic

dynamic equations as the simplest unilateral microdynamics that is exactly (analytically) solvable on the statistical level. The explicit expression of the microcrack distribution function makes it possible to investigate the relevance of some damage parameter and its respective evolution equation and the role of microscopic unilateral constraints.

2. Basic concepts of mesoscopic microcracking

In this section we give a short summary of the mesoscopic microcrack theory and the related equations governing the development of crack distribution function. For a more detailed treatment we refer to [35].

In a mesoscopic theory of a microcracked continuum our quantities are interpreted on the *directional space* that includes the $\mathbf{l}$ length of the microcracks and the $\mathbf{x}, t$ spacetime coordinates. The cracks are considered as penny shaped surfaces with area πl^2 and orientation $\mathbf{n}$. These two quantities characterizing the crack are put together as a vector $\mathbf{l} = l\mathbf{n}$, where l is the crack radius. The microstructure consists of this surface vector of the cracks. This vector is an axial one, which we do not consider as an antisymmetric tensor, but for the sake of convenience we use normal polar vectors with a symmetry requirement on their number density function:

$$N(\mathbf{l}, \mathbf{x}, t) = N(-\mathbf{l}, \mathbf{x}, t).$$

The basic equations of the mesoscopic theory are the mesoscopic balances of the fundamental physical quantities. Due to the enlarged configurational space, the mesoscopic velocities and the mesoscopic material current densities have six components and can be decomposed into a 'normal' part (with respect to the position variable) and a 'mesoscopic' or 'directional' part. The macroscopic number density of microcracks of any size and orientation at position $\mathbf{x}$ and time t, $\bar{N}$, can be introduced with an integration over the directional part of the mesoscopic space: $\bar{N}(\mathbf{x}, t) := \langle N(\mathbf{l}, \mathbf{x}, t)\rangle := \frac{1}{2}\int_{\mathbb{R}^3} N(\mathbf{l}, \mathbf{x}, t)\,\mathrm{d}V_l$. From the balance of the mesoscopic number density, the generalized continuity equation can be written as

$$\frac{\partial N}{\partial t} + \nabla \cdot (N\mathbf{v}) + \nabla_l \cdot (N\mathbf{v}_l) = \sigma_n. \tag{1}$$

Here $\mathbf{v}$ is the translational velocity, the space part of the mesoscopic velocity, and $\mathbf{v}_l$ is the 'directional' part. ∇_l denotes a derivation with respect to the variable $\mathbf{l}$ and as a part of the mesoscopic divergence represents the physical requirement that the number of crack with a given length at a given material element can be changed by changing the length of some smaller or larger cracks. The dot stands for the divergence, and the source term σ_n characterizes the creation and coalescence of microcracks.

We introduce a distribution function f as the probability density of a crack having a particular length and orientation as follows:

$$f(l, \mathbf{n}, \mathbf{x}, t) = \frac{N(l, \mathbf{n}, \mathbf{x}, t)}{\bar{N}(\mathbf{x}, t)}. \tag{2}$$

From the mesoscopic balance of number density follows the equation determining the time development of the crack distribution function is

$$\frac{\partial f}{\partial t} + \nabla \cdot (f\mathbf{v}) + \nabla_l \cdot (f\mathbf{v}_l) = \frac{-f}{\bar{N}}\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\bar{N} + \sigma_n. \tag{3}$$

In addition we have balance equations for the mesoscopic densities of all other extensive quantities, e.g., one can give the mesoscopic balances of mass, momentum, angular momentum and energy [36]. The macroscopic balances can be introduced by averaging the corresponding

physical quantities over the additional, enlarged part of their domain, using the previously introduced directional probability distribution function.

We are interested here in the evolution of crack lengths and damage is described in terms of the crack length distribution function. However, crack orientation was introduced as an additional mesoscopic variable together with crack length, and there may be an arbitrary distribution of crack orientations. Special cases of orientation distributions are: an isotropic distribution of all orientations, or, as the other extreme case, parallel orientations of all cracks. Very often we will meet cases in between: the cracks are not oriented exactly parallel, but there is a preferred orientation. In these cases the orientation distribution can be rotation symmetric (uniaxial) or not (biaxial). This classification in terms of the orientation distribution is analogous to liquid crystals [37, 38], where the mesoscopic variable is the orientation of elongated molecules. Here we want to consider only averages over all crack orientations. We introduce the crack length distribution function as the average over all crack orientations:

$$f(l, \mathbf{x}, t) = \frac{1}{4\pi} \int_{S^2} f(\mathbf{l}, \mathbf{x}, t)\, \mathrm{d}^2 n. \tag{4}$$

Averaging the differential equation (3) over orientations results in the equation of motion for the crack length distribution (11). (For more details see [39].) It is the same form of differential equation for any orientation distribution. Only the value of the crack length change velocity v_l depends on the orientation distribution via the effective stress acting on cracks of different orientations. Therefore we have only a parametric dependence of the differential equation to be solved for the crack length distribution on the orientation distribution, and the same form of solutions is expected for any orientation distribution.

The distribution functions are normalized:

$$\int_{S^2} \int_0^\infty f(\mathbf{l}, \mathbf{x}, t) l^2\, \mathrm{d}l\, \mathrm{d}^2 n = 1 \qquad \forall \mathbf{x}, t \tag{5}$$

and

$$\int_0^\infty f(l, \mathbf{x}, t) l^2\, \mathrm{d}l = 1 \qquad \forall \mathbf{x}, t. \tag{6}$$

3. Solution of the mesoscopic equations

There are essentially two methods to gain information from the basic equations of the mesoscopic theory. The first one, which was used mainly for liquid crystals, is to calculate a moment series expansion of the equations. Using this expansion we can introduce a hierarchy of macroscopic internal variables together with their evolution equations. This can also be useful in our special case; the nature of macroscopic damage parameters and the properties of their evolution equations is an old and discussed problem in continuum damage mechanics. However, now we will prefer another method: we will solve the mesoscopic equations in simple cases to get direct information on how the distribution function of microcracks evolves in time.

The system of mesoscopic balance equations given in the previous section is too general for most of the practical problems and materials in damage mechanics. In the case of brittle (rock or ceramic) material the following simplifications are suitable:

(1) The base material does not have an internal spin, that is a crack does not rotate independently of the base material.
(2) There are no couple forces and couple stresses.
(3) There are no external body forces ($\mathbf{f} = 0$).

(4) The material is in (local) mechanical equilibrium ($\dot{\mathbf{v}} = 0$).
(5) The translational velocity $\mathbf{v}$ does not depend on the crack size and orientation. It is the same for all cracks, namely it is equal to the velocity of the centre of mass of the surrounding continuum element ($\mathbf{v}(\mathbf{l}, \boldsymbol{x}, \mathbf{t}) = \bar{\mathbf{v}}(\boldsymbol{x}, t)$). This does not imply restrictions on the length change velocity.
(6) There is no creation of cracks. This is not a crucial restriction. For example, preexisting voids are considered as microcracks of very small size. In the progress of damage they grow, but their number is constant. (The voids in the initial stage are 'counted' as cracks.)
(7) There is no crack coalescence. Together with the previous assumption we postulate that there are no crack sources, i.e. the number of cracks is constant in time. Let us remark that the absence of coalescence does not mean the absence of interaction. Interaction can be considered in the microscopic dynamics as we will see below.

Let us remark that these assumptions are not necessary; more general cases can be treated in the frame of our model. For example, crack creation and coalescence can be considered through a suitable source term in (1). However, in this paper we maintain the theory as simple as possible to investigate some effects of microscopic unilateral dynamics.

All the previous assumptions are reasonable in the case of solid materials. Considering these assumptions we find that the balance of moment of momentum is trivial; simplifies to the condition of mechanical equilibrium:

$$\nabla \cdot \mathbf{t} = 0$$

where $\mathbf{t}$ is the stress. Moreover, at present we are not interested in the changes of the mass density and the produced energy during cracking; we will concentrate on the evolution of the crack distribution function.

As a result of conditions 5–7 the macroscopic number density of microcracks does not change in a material element (only the lengths can be changed) and there is no source term at the mesoscopic level. Therefore there is no source term in the differential equation for the crack distribution function (3). Due to condition 4 choosing a comoving frame results in a zero translational velocity, and we get the following simple equation

$$\frac{\partial f}{\partial t} + \nabla_l \cdot (f \mathbf{v}_l) = 0. \tag{7}$$

Let us mention here that in this case the equation above has a direct interpretation. From a pure statistical mechanical point of view this is a continuity equation for the probability density of microcracks having the 'microscopic' dynamical equation

$$\dot{\mathbf{l}} = \mathbf{v}_l(\mathbf{l}).$$

Here the dot denotes the substantial (comoving) time derivative. However, the special form of the differential equation for the distribution function depends on the equation for the length change velocity. This expression can involve a dependence on orientations and lengths of surrounding cracks and also several other forms of crack interaction. Equation (7) is a generalized form of the classical reversible Liouville equation of statistical mechanics, determining the dynamics of the distribution function f with the help of a known deterministic (!) microscopic dynamics [40]. From this observation it is clearly seen that in the simplest non-interacting, reversible case the mesoscopic theory is fully compatible with the basic statistical theories and continuum theories, too.

4. Rice–Griffith dynamics

As an example we will use one of the simplest, but well-known dynamics that was given by Rice from general thermodynamic considerations giving an irreversible thermodynamic

background to the energetic considerations of Griffith [41, 42]. The condition is derived for a single crack under uniform tensile loading parallel to the direction of the crack surface normal. The main assumption in the construction of the dynamics is that the Gibbs potential depends not only on the stress but also on the crack length in the following way:

$$G(\sigma, l) = \alpha l - \frac{\beta}{2}\sigma^2 l^2 - \frac{\gamma}{2}\sigma^2. \tag{8}$$

Here α is usually considered as a kind of surface energy, β is connected to the energy release rate in the case of simple geometric situations, γ is the elastic coefficient, l is the length of the crack and σ is the tensile stress parallel to the crack normal. Let us remark here that similar conditions can be given in the case of compressive stresses with modified material parameters (a compressive stress can result in a tensile force on a crack surface for certain crack orientations). The first term in (8) is usually interpreted as a surface energy of the crack, the second as the extra elastic energy of the surrounding continuum and the last one is the pure elastic term in this simple case. The sign and Legendre transformation conventions are according to Landau and Lifshitz [43].

The parameters α, β and γ and the coefficients in the Gibbs potential are effective parameters of the surrounding continuum, including other cracks.

From this Gibbs potential we can derive the entropy production σ_s due to the crack extension in the case of isothermal conditions

$$\sigma_s = \left(\frac{\partial^2 G}{\partial \sigma^2}\right)^{-1}\left(\frac{\partial G}{\partial \sigma} - \epsilon\right)\dot{\epsilon} - \frac{\partial G}{\partial l}\dot{l}.$$

It is easy to identify thermodynamic forces and currents in the expression. Moreover, we are interested only in the crack propagation effects, therefore we will suppose that during the propagation of the crack the material is in mechanical equilibrium, because the crack propagation is considerably slower than the deformation process. We conclude that

$$\epsilon = -\frac{\partial G}{\partial \sigma} = (\gamma + \beta l^2)\sigma \tag{9}$$

where ϵ is the deformation and we arrive at the evolution law of crack propagation

$$\dot{l} = -L\frac{\partial G}{\partial l} = -\alpha' + \beta' l \sigma^2.$$

Here L is a positive kinetic coefficient (according to the second law) and $\alpha' = L\alpha$, $\beta' = L\beta$. The equilibrium solution of this equation gives the celebrated Griffith condition [41]. After simple stability considerations one can see easily that the cracks start to grow if the right-hand side of the above differential equation is positive (in brittle materials e.g., rocks at room temperature) expressing the unilateral nature of crack propagation. If healing of the cracks is excluded, then the above evolution law can be written more exactly as

$$\dot{l} = \begin{cases} -\alpha' + \beta' l \sigma^2 & \text{if} \quad \alpha' \leqslant \beta' l \sigma^2 \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

This evolution equation will be a core for the mesoscopic dynamical investigation resulting in an explicit expression for the mesoscopic velocity. The above unilateral dynamics will be called Rice–Griffith dynamics.

Let us remark here that there is much more information that can be gained from these thermodynamic considerations. For example, as a condition of thermodynamic stability we get an upper limit for the stable crack length $\left(\frac{\gamma^2}{3\beta} > l^2\right)$ that one cannot read from the single (10) evolution equation without considering the mechanical interaction. The role of thermodynamic stability in the failure of real materials is investigated in more detail in

[18, 19, 44], where it is shown that localization and critical damage can be considered as the loss thermodynamic stability, and some empirical failure criteria (like Griffith) can be understood well from a thermodynamic point of view.

5. Solutions of the Liouville approximation

Let us consider now a sample with tensile hydrostatic loading. The mesoscopic variable in a given frame can be decomposed into the crack length and the crack orientation $\mathbf{l} = (l, \Theta)$. Similarly, the mesoscopic velocity can be decomposed into a length change part and an orientation change part, as $\mathbf{v}_l = (v_l, \omega)$. According to assumptions 1 and 4 the cracks cannot move independently of the surrounding continuum, and let the material be at rest $\mathbf{v} = \mathbf{0}$, therefore the orientational change velocity is zero ($\omega = 0$). Here we will investigate pure tensile loading conditions, when the crack surface normals are parallel to the applied loading. Due to our simplifications the role of crack orientation changes can be neglected in the development of the different order parameters [39].

In the following we will investigate constant loading rate conditions $\sigma = v_\sigma t$, $v_\sigma = \text{const}$, common in standard failure tests. Considering all of our assumptions regarding the cracks we can conclude that the crack density distribution depends only on time and the length of the cracks. The evolution equation (7) in spherical coordinates and in the mesoscopic space is the following,

$$\frac{\partial f}{\partial t}(t, l) = -\frac{1}{l^2}\frac{\partial l^2 v_l(l, t) f(t, l)}{\partial l} \tag{11}$$

where the mesoscopic speed, the crack length change velocity v_l, is given with the help of the right-hand side of equation (10), that is $v_l = \dot{l}$.

It is easy to check that the general solution of the first-order partial differential equation (11) with the microscopic dynamics (10) is the following:

$$f(l, t) = \begin{cases} l^{-2}\,\mathrm{e}^{-\frac{\beta' v_\sigma^2}{3}t^3} F\left(l\,\mathrm{e}^{-\frac{\beta' v_\sigma^2}{3}t^3} + \dfrac{\alpha'}{\left(9\beta' v_\sigma^2\right)^{1/3}}\Gamma\left(1/3, 0, \dfrac{\beta' v_\sigma^2}{3}t^3\right)\right) & \text{if} \quad \alpha' \leqslant \beta' v_\sigma^2 l t^2 \\ f(l, 0) & \text{otherwise.} \end{cases}$$

Here Γ is the generalized incomplete Gamma function defined by $\Gamma(a, b, c) = \Gamma_i(a, c) - \Gamma_i(b, c)$, where Γ_i denotes the incomplete Gamma function. F is an arbitrary function to be determined from the initial conditions. One can observe that there is no stationary solution of (11), but the crack distribution with a finite support does not change under a definite loading. A solution with arbitrary initial conditions cannot be given analytically because of the non-healing microcracks, the non-differentiable right-hand side of (10). As a consequence of this non-differentiability, the solutions of the partial differential equation can be split into two parts connected by a moving boundary. Smaller and smaller cracks start to grow arriving at the moving boundary. The arbitrary function F should be determined in such a way that the final distribution function will be continuous.

In the following we introduce dimensionless variables, measuring the time in units of $\hat{t} = v_\sigma^{-2/3}\beta^{-1/3}$, the length in $\hat{l} = \alpha v_\sigma^{-2/3}\beta^{-1/3}$ and the mass $\hat{m} = \alpha v_\sigma^{-5/3}\beta^{-4/3}$ as the crack length l, the stress σ, the stress change velocity v_σ and the material parameters α' and β' are measured in SI units. The dimensionless microdynamic equation (10) does not contain material parameters (e.g., can be written as $v_\sigma = 1, \alpha' = 1$ and $\beta' = 1$ in (10)). In the following all the variables are understood as dimensionless.

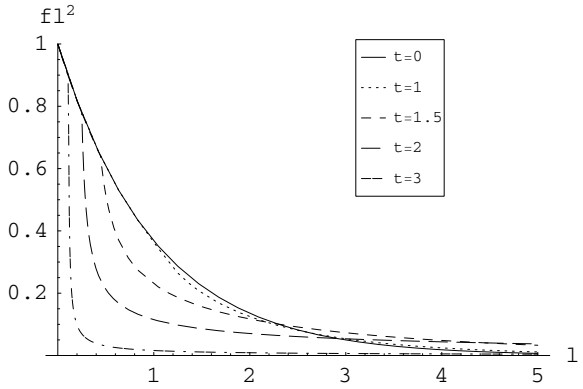

Figure 1. The crack length distribution function at different instants. Exponential initial condition.

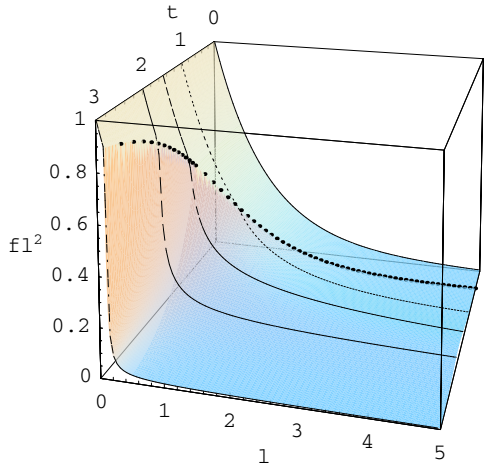

Figure 2. The time development of the crack length probability distribution. Exponential initial condition.

The solution of this equation is calculated numerically in the case of two different initial length distribution functions. First, an exponential initial crack length distribution function is considered,

$$f(l,0) = l^{-2}\frac{\mathrm{e}^{-\frac{l}{\delta}}}{\delta}.$$

You can see the solution with the parameter value $\delta = 1$ at different instants in figure 1. On the vertical axis the function $l^2 f(l,t)$ is drawn (because f multiplied with the spherical weight has the direct probability interpretation). One can observe that the breaking point of the initial exponential distribution moves to the left, according to the Rice–Griffith dynamics.

In figure 2 we can observe the full time and crack length development of the distribution function. The projection of the thick dotted line, where the surface is broken, onto the l–t plane gives the Griffith condition, considering the time dependence of stress. The other lines show the distribution function at different instants.

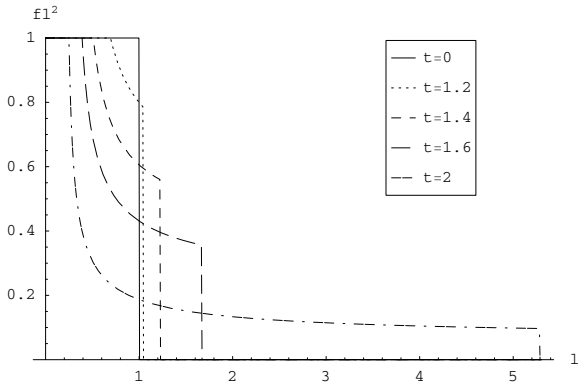

Figure 3. The crack length distribution function at different instants. Stepwise initial condition.

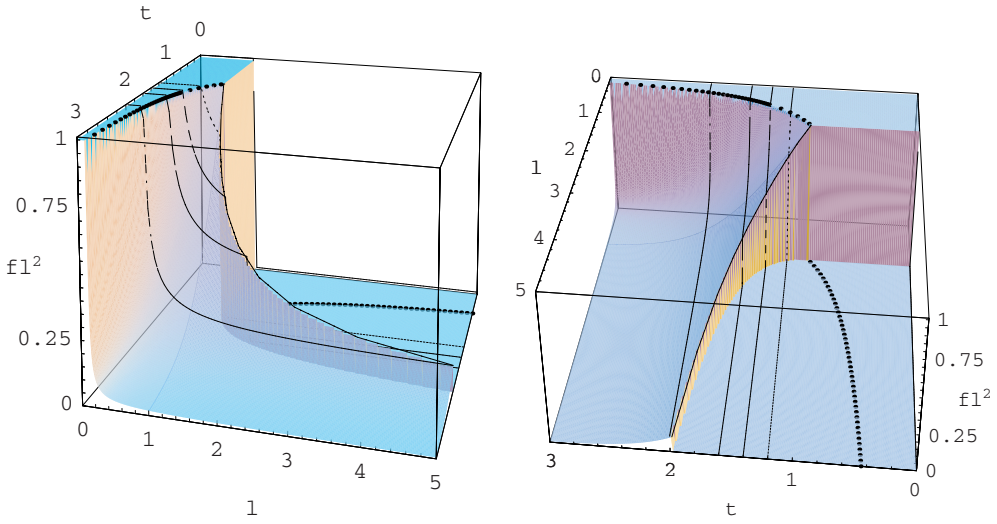

Figure 4. Different views of the time development of the crack length probability distribution. Stepwise initial condition.

As a more realistic example, which considers an upper limit of the possible crack lengths, we treat a bounded uniform initial crack distribution function, too:

$$f(l,0) = \begin{cases} \dfrac{l^{-2}}{l_c} & \text{if} \quad l < l_c \\ 0 & \text{otherwise.} \end{cases}$$

In the following we choose $l_c = 1$. In figure 3 one can see the crack length distribution at different instants. Figure 4 shows the time and length dependence together. The thick dotted line and its projection onto the l–t plane is the Griffith condition, which is more apparent in the backward view. The thin lines are the crack distribution functions at the same instants as in figure 3.

Here the resulting solution is discontinuous, due to the discontinuous initial condition. The initial discontinuity is conserved but move towards to larger crack sizes with increasing time.

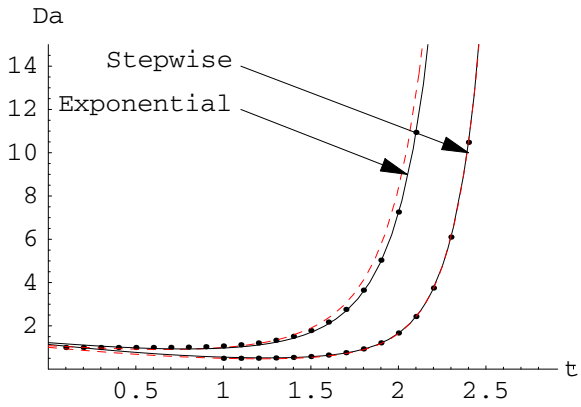

Figure 5. Time dependence of damage in the case of exponential and stepwise initial distributions. The dots are related to the averaged distribution function and the dashed and continuous lines are the lines calculated from the macrodynamic equations.

In [39] we suggest several possible macroscopic internal variables from a mesoscopic point of view. Different order and damage parameters are introduced and the possible advantages and disadvantages are discussed. Here we will investigate one of the most remarkable candidates, the average crack length, and calculate its time dependence in the case of the two treated initial crack length distributions. The average crack length will be called in the following *damage* and defined as

$$D(t) = \langle l \rangle = \int_0^\infty l f(l,t) l^2 \,\mathrm{d}l = \int_0^\infty l^3 f(l,t)\,\mathrm{d}l.$$

The notion of an order parameter is chosen in analogy with the mesoscopic theory of liquid crystals, where the analogously defined orientational moments of the distribution function, the alignment tensors, are the order parameters in a Ginzburg–Landau theory. There the gradient of the order parameter contributes to the free energy density in the form of the Frank elastic energy. Here, in the mesoscopic theory of damage, we did not take into account any spatial gradients at all, and therefore there is no direct comparison to a Ginzburg–Landau theory. Definitely the damage parameter introduced here is different from the order parameter, the displacement field in phase field models. The spatial structure of the crack is out of the scope of the mesoscopic theory, and we are dealing only with microcracks smaller than the typical dimension of the continuum element. On the other hand, the onset of damage has been interpreted as a loss of thermodynamic stability with the help of a free energy depending on a damage parameter density [18]. The inclusion of spatial inhomogeneities of the damage parameter, and the formulation of an extension of a Ginzburg–Landau-type theory of damage (with a mesoscopic background) is an outstanding problem for the future.

After calculating the time dependence of the damage (the dots in figures 5 and 6) we can observe that time development is similar to the solution of the microscopic dynamic equation. Therefore we can investigate whether the damage development is determined by a Rice–Griffith dynamical equation. More properly we suppose that the evolution equation that determines the damage dynamics is the following:

$$\dot{D}(t) = -a + bt^2 D(t). \tag{12}$$

One can get these equations by averaging the mesoscopic dynamic equations without considering the unilateral property of the microdynamics. The above equation is the only reasonable candidate to describe the macroscopic dynamics of the averaged crack length.

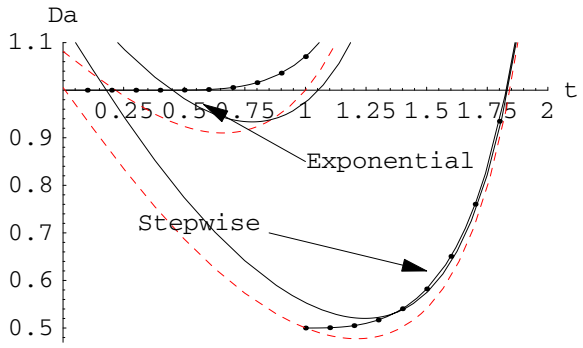

Figure 6. Time dependence of damage in the case of exponential and stepwise initial distributions (enlarged initial part of figure 5).

To compare the exact and the suggested macrodynamics we fitted the solutions of the above equation to the exact solution to determine the parameters of the damage dynamics and the dependence on the initial conditions. In both cases we considered the possibility that the damage starts from the initial value determined from the initial crack distribution (dashed lines in figures 5 and 6) or the initial damage belongs to the parameters to be determined (continuous lines in figures 5 and 6). The results are summarized in the following table.

	t_0	$D(t_0)$	a	b
Exponential	0	1	0.395	1
	0	1.2226	0.5681	1
Stepwise	1	0.5	0.7006	1
	1	0.5534	0.8061	1

The parameters were determined by an asymptotic approximation method considering more and more sample points. We can observe that the $\beta' = 1$ parameter of the microscopic crack dynamics is the same as the analogous b in the case of damage dynamics; however, the $\alpha' = 1$ and a parameters (the 'surface energy' times the dynamic coefficient) are smaller for both initial distributions. The averaged macroscopic equations especially in the case of fitted initial conditions give a remarkably good approximation for larger times as one can see, e.g., in figure 5. However, at the beginning when the number of 'frozen' cracks is large, the damage developments are qualitatively different (see figure 6). One can observe a kind of 'pseudo healing', the damage curves calculated from the macroscopic considerations start to decrease, instead of the expected increase. This effect is due to the unilateral microdynamics. The averaged distribution function is monotonically increasing but the suggested approximate macrodynamic equation at the beginning starts to decrease. The decrease depends on the initial distribution. The initial part of the curves related to the stepwise distribution is an extrapolation in both figures to show that a unilateral macrodynamic equation would not improve the situation.

6. Discussion and final remarks

In this paper the equations of mesoscopic microcracking were solved in the case of simple cracking processes. We have seen that under reasonable assumptions the seemingly complex

mesoscopic balances are reduced to a solvable first-order partial differential equation giving a kind of Liouville approximation of the mesoscopic theory. There we could get the mesoscopic velocities introducing some constitutive assumptions based on known single microcrack dynamics. The partial differential equation has an independent statistical physical interpretation and can also be useful to describe the evolution of microstructure without the underlying mesoscopic theory. Moreover, it gives a sure basis of the internal variable calculations, giving particular interpretations of the different internal variables and clues regarding their dynamics, the changes of the elastic moduli or the stress distribution without any further physical assumptions.

The development of the average crack length was investigated in detail and we have got a macroscopic dynamics that is almost the same as for single cracks with the remarkable property that the corresponding parameters of the damage evolution equation strongly depend on the initial crack distribution. Moreover, the statistical calculations resulted in a new qualitative phenomenon: the average crack length was decreasing at the beginning, seemingly contradicting the unilateral microdynamics of the microcracks. Those calculations give some insight into the applicability of internal variable approaches in nonequilibrium situations. The unilateral microscopic conditions alone (without quenched disorder) resulted in a macroscopic dynamics of the damage (internal variables) that proved to be a good approximation of the microscopic dynamics, but with parameters strongly depending on the initial crack distribution.

Our assumptions regarding the microscopic, single crack propagation evolution equations are simplified but can be extended to far more difficult cases. There is no conceptual difficulty in considering more general loading conditions, microscopic dynamics with terminal crack velocity, etc. Microdynamic equations that incorporate terminal velocity and unilaterality together can be particularly interesting. However, more involved microdynamics can overshadow the reasons behind the observed properties of macroscopic damage.

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